# Stochastic Reduction in Nonlinear Quantum Mechanics

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Stochastic extensions of the Schrödinger equation have attracted attention recently as plausible models for state reduction in quantum mechanics. Here we formulate a general approach to stochastic Schrödinger dynamics in the case of a nonlinear state space of the type proposed by Kibble. We derive a number of new identities for observables in the nonlinear theory, and establish general criteria on the curvature of the state space sufficient to ensure collapse of the wave function.

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A generalisation of quantum mechanics was considered by Mielnik [1], who introduced the notion of nonlinear observables. Two alternative extensions of the standard quantum theory were then proposed by Kibble [2]. The first alternative is based on the phase space formulation of quantum mechanics. Here, if we work with the space of rays through the origin of Hilbert space, then the Schrödinger equation reduces to Hamilton's equation of classical mechanics [3], except that the quantum Hamiltonian is of a special restricted form. Thus a natural generalisation is to remove this constraint. When such trajectories are lifted from the space of rays to Hilbert space, we obtain nonlinear wave equations.

The general properties of nonlinear observables were subsequently analysed in detail by Weinberg [4]. Following this, it was pointed out by Gisin [5] that the evolution of the density matrix is not autonomous in the nonlinear mechanics of [2,4], and that this may be physically undesirable. However, it was also indicated in [5] that there is another type of nonlinear quantum dynamics for which the evolution of the density matrix is autonomous, and a number of desirable features of linear evolution are extended in a natural way. This is the stochastic dynamics developed by Pearle and others [6]. These dynamics are of significance because they exhibit natural reductive properties: starting from a given initial state, the system evolves stochastically in such a way to ensure collapse to an eigenstate of one or more designated observables.

Kibble's second alternative for a nonlinear quantum theory is in essence to consider a general Kähler manifold as the phase space of quantum mechanics, instead of the space of rays. The idea is that, in the presence of interactions, the states accessible to a quantum system constitute a curved space $\mathfrak{M}$ which has the structure of a complex manifold endowed with a compatible symplectic structure. The dynamics of the state are then governed by a Hamiltonian flow which is also an isometry.

In the present article, we consider stochastic state reduction models within the framework of Kibble's second theory. The advantage of a stochastic dynamics in this context is that it leads to a probabilistic interpretation, a feature hitherto missing in the nonlinear theory. Remarkably, many of the key features of the basic stochastic reduction models carry through to a fully nonlinear state space. The main results are to determine general criteria sufficient to ensure state reduction in the nonlinear theory, and to express these criteria directly in terms of geometrical features of the state manifold. Thus it is the geometry of the quantum state manifold that determines whether reduction takes place, and if so, how rapidly.

After introducing the relevant state space geometry and elements of stochastic calculus on manifolds, a number of identities concerning the properties of quantum observables are established in Lemmas 1-5. These results are then applied to formulate general theorems governing reduction processes on nonlinear state spaces.

Let us first recall briefly the phase space formulation of quantum theory. We consider a finite dimensional complex Hilbert space $\mathcal{H}$ of which a typical element is denoted $\psi^\alpha$ ($\alpha = 0, 1, \cdots, n$). Given the Hamiltonian operator $H^\alpha_\beta$, the dynamics of the state is determined by the Schrödinger equation $i\hbar\partial_t\psi^\alpha = H^\alpha_\beta\psi^\beta$. The expectation $F^\alpha_\beta\bar{\psi}_\alpha\psi^\beta/\bar{\psi}_\gamma\psi^\gamma$ of an observable $F^\alpha_\beta$ in the state $\psi^\alpha$ is invariant under the scale transformations $\psi^\alpha \to \lambda\psi^\alpha$ ($\lambda \in \mathbb{C} - \{0\}$). Hence we can work with the space of equivalent classes of state vectors modulo such transformations, i.e., the complex projective space $P^n$.

We regard $P^n$ as a real manifold $\Gamma$ of dimension $2n$. It is known that $\Gamma$ has a natural symplectic structure $\omega_{ab}$, as well as a Riemannian structure given by the Fubini-Study metric $g_{ab}$. These two structures are compatible in the sense that there exists an integrable complex structure $J^a_{\ b}$ on $\Gamma$, satisfying $J^a_{\ c}J^c_{\ b} = -\delta^a_{\ b}$, such that $\nabla_a J^b_{\ c} = 0$ and $g^{ac}\omega_{cb} = J^a_{\ b}$, where $\nabla_a$ is the covariant derivative associated with $g_{ab}$, and $g_{ac}g^{cb} = \delta_a^{\ b}$. We use Roman indices $(a, b, \cdots)$ for tensorial operations on $\Gamma$. The compatibility conditions make $\Gamma$ a Kähler manifold.

The special feature that identifies $\Gamma$ as the quantum phase space is that the Schrödinger equation can be expressed in the Hamiltonian form $\hbar dx^a/dt = 2\omega^{ab}\nabla_b H(x)$. Here $\omega^{ab} = g^{ac}g^{bd}\omega_{cd}$ and $H(x)$ is the Hamiltonian function on $\Gamma$, given by the expectation of the operator $H^\alpha_\beta$ in the equivalence class of state vectors corresponding to

the point $x \in \Gamma$. The vector $Z^a = \omega^{ab}\nabla_b H$ tangent to the Schrödinger trajectory satisfies the Killing equation $\nabla_{(a}Z_{b)} = 0$ iff $H(x)$ is the expectation of a quantum observable in the state $x$. Therefore, the Schrödinger evolution preserves the distance, and hence the transition probability, between any given two states. More generally, any isometry of $\Gamma$ is a Hamiltonian flow associated with a quantum observable. The energy eigenstates are the fixed points of the flow, at which $\nabla_a H = 0$. The first alternative of Kibble is to replace the observable $H(x)$ by a general function on $\Gamma$. Then the resulting trajectories are Hamiltonian but no longer Killing, and the implied dynamics on $\mathcal{H}$ is governed by a nonlinear wave equation. This is not the generalisation we consider here.

For the consideration of Kibble's second alternative, it will be useful first to develop a differential geometric framework for the standard operations of quantum mechanics (we set $\hbar = 1$). If $F(x)$ and $G(x)$ are observables, the expectation of their commutator is also an observable, given by the Poisson bracket $2\omega^{ab}\nabla_a F\nabla_b G$. Then if $x_t$ is a Schrödinger trajectory and $F_t = F(x_t)$, it follows that $dF_t = 2\omega^{ab}\nabla_a F\nabla_b H dt$, where $H$ is the Hamiltonian. This tells us how the expectation of $F$ changes along the flow generated by $H$. For any observable $F(x)$, we define the associated dispersion by

$$V^F = g^{ab}\nabla_a F\nabla_b F. \tag{1}$$

In the linear theory $V^F(x)$ is the squared uncertainty of $F$ in the state $x$. As a consequence of the inequality $(g_{ab}X^aX^b)(g_{ab}Y^aY^b) \geq (g_{ab}X^aY^b)^2 + (\omega_{ab}X^aY^b)^2$, which holds for all $X^a$ and $Y^a$, we obtain the Heisenberg relation $V^FV^G \geq (\omega^{ab}\nabla_a F\nabla_b G)^2$ if we set $X^a = \omega^{ab}\nabla_b F$ and $Y^a = \omega^{ab}\nabla_b G$.

Kibble's second alternative for a nonlinear quantum theory is to let the state space be a general Kähler manifold $\mathfrak{M}$, with metric $g_{ab}$, symplectic structure $\omega_{ab}$, and complex structure $J_a{}^b$. In the nonlinear theory we say that a real function $F(x)$ on $\mathfrak{M}$ is an observable iff the corresponding Hamiltonian vector field $X^a = \omega^{ab}\nabla_b F$ is an isometry. This agrees with the usual characterisation of observables when $\mathfrak{M} = \Gamma$. We note that if $\mathfrak{M}$ is compact and has vanishing first Betti number, then any Killing field on $\mathfrak{M}$ is Hamiltonian, with at least two distinct eigenstates [7]. As in the linear theory, we interpret $F(x)$ as the expectation of the result of a measurement of the given observable in the state $x \in \mathfrak{M}$.

**Lemma 1**. *If $F(x)$ and $G(x)$ are observables, then their commutator is an observable.*

The proof is as follows. If $X^a$ and $Y^a$ are Hamiltonian flows, then their Lie bracket is $X^b\nabla_b Y^a - Y^b\nabla_b X^a = \omega^{ab}\nabla_b(\omega_{cd}X^cY^d)$. If $F(x)$ and $G(x)$ are generators of $X^a$ and $Y^a$, then $\omega_{cd}X^cY^d = \omega^{cd}\nabla_c F\nabla_d G$. Furthermore, if $X^a$ and $Y^a$ are Killing, so is their Lie bracket. Therefore, the Hamiltonian flow generated by the commutator of $F(x)$ and $G(x)$ is Killing. As a consequence we obtain also the following nonlinear generalisation of an identity due to Adler and Horwitz [8]:

**Lemma 2**. *If $F(x)$ and $G(x)$ are observables, then $\nabla_b F\nabla^b\nabla^a G - \nabla_b G\nabla^b\nabla^a F = \omega^{ab}\nabla_b(\omega^{cd}\nabla_c F\nabla_d G)$.*

As in the linear theory, the eigenstates of an observable $F(x)$ are the points of $\mathfrak{M}$ at which $\nabla_a F = 0$. The value of $F(x)$ at a critical point is the corresponding eigenvalue. In the nonlinear theory $V^F(x)$ does not in general have an interpretation as a moment, but nevertheless remains a measure of the dispersion of $F(x)$ in the given state. In particular, the Heisenberg relation holds.

The Schrödinger trajectories in the nonlinear theory are generated by a Hamiltonian $H(x)$, which we assume to be an observable. The following result shows that for any observable commuting with the Hamiltonian, its dispersion is constant along the Schrödinger trajectory. This fact will be used later to derive the stochastic dynamics of the energy dispersion.

**Lemma 3**. *If $F$ is an observable that commutes with the Hamiltonian $H$, then $\omega^{ab}\nabla_a H\nabla_b V^F = 0$.*

The proof is as follows. Equation (1) implies that $\omega^{ab}\nabla_a H\nabla_b V^F = 2\omega^{ab}\nabla_a H\nabla_b\nabla_c F\nabla^c F$, and thus

$$\begin{aligned}\omega^{ab}\nabla_a H\nabla_b V^F = {} & 2\nabla_c(\omega^{ab}\nabla_a H\nabla_b F)\nabla^c F \\ & -2\nabla_c(\omega^{ab}\nabla_a H)\nabla_b F\nabla^c F. \end{aligned} \tag{2}$$

The first term on the right vanishes because $H$ and $F$ commute, whereas the second term vanishes on account of the Killing equation satisfied by $Z^a$.

To proceed further we now introduce briefly the elements of stochastic differential geometry. The basic process we consider is the Wiener process $W_t$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega$ is the sample space, $\mathcal{F}$ denotes the filtration, and $\mathbf{P}$ is the probability measure. We say that $W_t$ is a Wiener process if it is continuous, $W_0 = 0$, $W_t - W_s$ $(0 \leq s < t)$ is independent of the information $\mathcal{F}_s$ up to time $s$, and $W_t - W_s$ is normally distributed with mean zero and variance $t - s$. A process $\sigma_t$ is said to be adapted to the filtration $\mathcal{F}_t$ generated by $W_t$ if its random value at time $t$ is determined by the history of $W_t$ up to that time. If $\sigma_t$ is $\mathcal{F}_t$-adapted, then the stochastic integral $M_t = \int_0^t \sigma_s dW_s$ exists, provided $\sigma_t$ is almost surely square-integrable. If the variance of $M_t$ exists, then $M_t$ satisfies the martingale conditions $\mathbb{E}[|M_t|] < \infty$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, where $\mathbb{E}[-]$ denotes expectation with respect to $\mathbf{P}$. The second condition implies that, given the history of the Wiener process up to time $s$, the expectation of $M_t$ for $t \geq s$ is given by its value at $s$. The variance of $M_t$ is determined by the Ito isometry $\mathbb{E}[M_t^2] = \mathbb{E}[\int_0^t \sigma_s^2 ds]$.

A general Ito process is defined by the integral $x_t = x_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$, where the adapted processes $\mu_t$ and $\sigma_t$ are called the drift and the volatility of $x_t$. A convenient way to express this is to write $dx_t = \mu_t dt + \sigma_t dW_t$. In the special case $\mu_t = \mu(x_t)$ and $\sigma_t = \sigma(x_t)$, where $\mu(x)$ and $\sigma(x)$ are prescribed functions, the process $x_t$ is said to be a diffusion.

This analysis can be generalised to the case of a diffusion $x_t$ taking values on a manifold $\mathfrak{M}$, driven by a standard $m$-dimensional Wiener process $W_t^i$ $(i =$

$1, 2, \cdots, m)$. Let $\nabla_a$ be a torsion-free connection on $\mathfrak{M}$, and suppose $\mu^a(x)$ and $\sigma^a_i(x)$ are $m+1$ vector fields on $\mathfrak{M}$. Then for the general diffusion on $\mathfrak{M}$ we have the stochastic equation $dx^a = \mu^a dt + \sigma^a_i dW^i_t$, where $dx^a$ is the covariant Ito differential. The quadratic relation $dx^a dx^b = h^{ab}dt$, where $h^{ab} = \sigma^a_i \sigma^{bi}$, follows from the identities $dt^2 = 0$, $dt dW^i_t = 0$, and $dW^i_t dW^j_t = \delta^{ij}dt$. If we choose $\mu^a = 2\omega^{ab}\nabla_b H$, $\sigma^a_i = 0$, and $\mathfrak{M} = \Gamma$, then $x_t$ reduces to the Schrödinger evolution.

For any smooth function $\phi(x)$ on $\mathfrak{M}$ we define the process $\phi_t = \phi(x_t)$, and Ito's formula takes the form $d\phi_t = (\nabla_a\phi)dx^a + \frac{1}{2}(\nabla_a\nabla_b\phi)dx^a dx^b$, or, more explicitly,

$$d\phi = \left(\mu^a\nabla_a\phi + \tfrac{1}{2}h^{ab}\nabla_a\nabla_b\phi\right)dt + \sigma^a_i\nabla_a\phi dW^i_t. \qquad (3)$$

The probability law for $x_t$ is characterised by a density function $\rho(x,t)$ which satisfies the Fokker-Planck equation $\partial\rho/\partial t = -\nabla_a(\mu^a\rho) + \frac{1}{2}\nabla_a\nabla_b(h^{ab}\rho)$. Such a diffusion is nondegenerate if $h^{ab}$ is of maximal rank. In particular, if $g_{ab}$ is a Riemannian metric on $\mathfrak{M}$ and $\nabla_a$ is the associated Levi-Civita connection, then if $h^{ab} = \sigma^2 g^{ab}$, the process $x_t$ is a Brownian motion with drift on $\mathfrak{M}$, with volatility parameter $\sigma$. If $h^{ab}$ is not of maximal rank, then the diffusion is degenerate, which is the case of relevance to the present consideration.

Our intention is to generalise the Schrödinger dynamics to a stochastic process on a nonlinear quantum state manifold $\mathfrak{M}$. Specifically, we consider the stochastic reduction model of Hughston [9], for which the dynamical trajectories are governed by the following stochastic differential equation:

$$dx^a_t = \left(2\omega^{ab}\nabla_b H - \tfrac{1}{4}\sigma^2\nabla^a V^H\right)dt + \sigma\nabla^a H dW_t. \qquad (4)$$

When $\mathfrak{M}$ is the state space $\Gamma$ of linear quantum mechanics, then (4) has the following interpretation. The first term in the drift generates the unitary part of the evolution, while the second term creates a tendency for the system to evolve to a state of lower energy variance. The volatility term is given by the gradient of the Hamiltonian, and generates fluctuations that die down as the system approaches an eigenstate. The parameter $\sigma$ controls the magnitude of the fluctuations. Starting from any initial state, the state vector collapses to an energy eigenstate, with collapse probability given by the Dirac transition probability. Furthermore, if the evolution of the density function $\rho(x,t)$ associated with the process (4) is lifted to $\mathcal{H}$, we recover the Lindblad form of the density matrix dynamics [5,10].

In the case of a general nonlinear quantum phase space $\mathfrak{M}$, the stochastic process (4) can be carried over directly, and we obtain the following characterisation of the dynamics of the energy:

**Theorem 1**. *The Hamiltonian process $H_t = H(x_t)$ is a martingale, given by the stochastic integral $H_t = H_0 + \sigma\int_0^t V_s dW_s$, where $V_t = V^H(x_t)$.*

Therefore, under the stochastic dynamics (4), the energy of the system is weakly conserved in the sense that the expectation of the value of the Hamiltonian at any future time is given by the initial value $H_0$. In particular, if reduction occurs, the martingale property ensures that the expectation of the terminal value of the energy, which is given by the sum of the energy eigenvalues weighted by the associated transition probabilities, is $H_0$. This in turn justifies the interpretation of $H(x)$ as the expectation of the energy in the given state $x \in \mathfrak{M}$. The proof of Theorem 1 follows from an application of Ito's formula (3). The fact that $H_t$ is a martingale does not imply that reduction occurs. For a reduction to energy eigenstates, we require $\lim_{t\to\infty} V_t = 0$. To determine the circumstances under which this occurs, we consider the dynamics of $V_t$.

**Lemma 4**. *The process $V_t = \nabla_a H\nabla^a H$ satisfies*

$$dV_t = \sigma^2(\nabla^a H\nabla^b H\nabla^c H\nabla_a\nabla_b\nabla_c H)dt + \sigma(\nabla^a H\nabla_a V)dW_t. \qquad (5)$$

The proof is as follows. We note that according to Ito's formula (3) we have

$$dV_t = \left(2\omega^{ab}\nabla_a V\nabla_b H - \tfrac{1}{4}\sigma^2\nabla_a V\nabla^a V + \tfrac{1}{2}\sigma^2\nabla^a H\nabla^b H\nabla_a\nabla_b V\right)dt + \sigma\nabla^a H\nabla_a V dW_t. \qquad (6)$$

The first term in the drift vanishes by Lemma 3. The remaining two terms in the drift combine to yield (5), because $\nabla^a H\nabla^b H\nabla_a\nabla_b V = 2\nabla^a H\nabla^b H\nabla^c H\nabla_a\nabla_b\nabla_c H + \frac{1}{2}\nabla_a V\nabla^a V$.

For state reduction, we need to show that the drift of $V_t$ is negative. To obtain a suitable criterion for this we proceed as follows. If $X_a$ is a Killing field, the cyclic identity $\nabla_{[a}\nabla_b X_{c]} = 0$ implies that $\nabla_c\nabla_a X_b = R_{abc}{}^d X_d$, where the Riemann tensor is defined by $\nabla_a\nabla_b A_c - \nabla_b\nabla_a A_c = -R_{abc}{}^d A_d$ for any vector field $A_a$. As shown in [11] we thus have:

**Lemma 5**. *If $F(x)$ is an observable on $\mathfrak{M}$, then $\nabla_a\nabla_b\nabla_c F = -J_b{}^p R_{apc}{}^q J_q{}^r\nabla_r F$.*

Next, we define the holomorphic sectional curvature $\mathcal{K}_H$ of the Kähler manifold $\mathfrak{M}$ with respect to the $J$-invariant plane $\nabla^{[a}H J^{b]}_c\nabla^c H$ by the formula

$$\mathcal{K}_H = -\frac{R_{apbq}J^p_c J^q_d\nabla^a H\nabla^b H\nabla^c H\nabla^d H}{(\nabla_a H\nabla^a H)^2}. \qquad (7)$$

The meaning of $\mathcal{K}_H$ is as follows. At each point $x \in \mathfrak{M}$ we consider the tangent plane spanned by vectors $\nabla^a H$ and $J^b_c\nabla^c H$. Then the totality of the geodesic curves tangent to this plane at $x$ forms a two dimensional surface in $\mathfrak{M}$, and the Gauss curvature of this surface at $x$ is $\mathcal{K}_H$.

**Theorem 2**. *If $\mathcal{K}_H > 0$, then $V^H$ is a supermartingale and (4) is a reduction process.*

The proof is by virtue of Lemmas 4 and 5. Writing $\mathcal{K}_t = \mathcal{K}_H(x_t)$, we deduce that

$$V_t = V_0 - \sigma^2\int_0^t \mathcal{K}_s V_s^2 ds + \sigma\int_0^t \nabla_a H\nabla^a V dW_s, \qquad (8)$$

and thus $\mathbb{E}[V_t|\mathcal{F}_s] \leq V_s$, the supermartingale condition. In particular, if we write $\bar{V}_t = \mathbb{E}[V_t]$ then it follows from (8) that $d\bar{V}_t/dt = -\kappa\sigma^2\bar{V}_t^2(1+\eta_t)$, where $\bar{V}_t^2\eta_t = \mathbb{E}[(V_t-\bar{V}_t)^2] + \kappa^{-1}\mathbb{E}[(\mathcal{K}_t-\kappa)V_t^2]$, and $\kappa = \inf_{\mathfrak{M}}\mathcal{K}_H$. Integrating, we obtain $\bar{V}_t = V_0/(1+\kappa\sigma^2 V_0(t+\xi_t))$, where $\xi_t = \int_0^t \eta_s ds$. The Hamiltonian sectional curvature is positive iff $\kappa > 0$, in which case $\xi_t \geq 0$. It follows that $\bar{V}_t \leq V_0/(1+\kappa\sigma^2 V_0 t)$, and thus $\lim_{t\to\infty} V_t = 0$ almost surely. Therefore, wave function collapse on the nonlinear state space is guaranteed if the holomorphic sectional curvature is positive. The characteristic reduction timescale is $\tau = (\kappa\sigma^2 V_0)^{-1}$, and for $t \gg \tau$ the uncertainty is reduced to a fraction of its initial value.

Now we are in a position to determine the relationship between the initial energy uncertainty $V_0 = V^H(x_0)$ and the terminal variance of the energy as a result of a reduction. It follows from Theorem 1, together with the Ito isometry, that $\mathbb{E}\left[(H_t-H_0)^2\right] = \sigma^2\bar{Q}_t$, where $H_0 = \mathbb{E}[H_t]$, $\bar{Q}_t = \mathbb{E}[Q_t]$, and $Q_t = \int_0^t V_s^2 ds$. On the other hand, if $\kappa > 0$ then from (8) we obtain $\bar{V}_t \leq V_0 - \kappa\sigma^2\bar{Q}_t$. Furthermore, if $\lambda = \sup_{\mathfrak{M}}\mathcal{K}_H$ and $\lambda > 0$, then (8) implies that $\bar{V}_t \geq V_0 - \lambda\sigma^2\bar{Q}_t$. Therefore, by Theorem 2, if $\mathcal{K}_H > 0$, we obtain the following bounds for the terminal energy variance:

$$\frac{V_0}{\kappa} \geq \lim_{t\to\infty}\mathbb{E}\left[(H_t-H_0)^2\right] \geq \frac{V_0}{\lambda}. \tag{9}$$

In particular, if $\mathfrak{M} = \Gamma$, it follows from the relation $R_{abcd} = -\frac{1}{4}(g_{ac}g_{bd} - g_{bc}g_{ad} + \omega_{ac}\omega_{bd} - \omega_{bc}\omega_{ad} + 2\omega_{ab}\omega_{cd})$ that $\mathcal{K}_H = 1$ and $V_0$ is the terminal energy dispersion.

An important issue in the consideration of state reduction processes is whether an energy-based dynamics suffices. To address this issue we examine the processes induced by (4) for observables other than $H$.

**Theorem 3**. *If an observable $F$ commutes with the Hamiltonian, then under the stochastic Schrödinger dynamics* (4) *the process $F_t = F(x_t)$ is a martingale.*

The proof follows as a consequence of Ito's lemma with an application of Lemmas 2 and 3. Theorem 3 generalises a result of [8] obtained in the case $\mathfrak{M} = \Gamma$. To determine whether the system necessarily collapses to an eigenstate of $F$ under (4) we require the concept of holomorphic bisectional curvature [12].

**Theorem 4**. *If the observable $F$ commutes with the Hamiltonian, then the stochastic equation for $V_t^F$ is*

$$dV_t^F = -\sigma^2\mathcal{K}_{FH}V^F V^H dt + \sigma\nabla_a H\nabla^a V^F dW_t, \tag{10}$$

*where the holomorphic bisectional curvature of $\mathfrak{M}$ with respect to the $J$-invariant planes determined by $F$ and $H$ is defined by*

$$\mathcal{K}_{FH} = -\frac{R_{apbq}J^p{}_c J^q{}_d \nabla^a F\nabla^b H\nabla^c F\nabla^d H}{\nabla_a F\nabla^a F\nabla_b H\nabla^b H}. \tag{11}$$

To prove this result, we use Ito's formula (3) to obtain

$$dV_t^F = \tfrac{1}{2}\sigma^2\left(\nabla^a H\nabla^b H\nabla_a\nabla_b V^F - \nabla^a\nabla^b H\nabla_a H\nabla_b V^F\right)dt + \sigma\nabla^a H\nabla_a V^F dW_t. \tag{12}$$

Then, by use of Lemmas 2, 3, 5, a calculation shows that the two terms in the drift combine to give (10). As a consequence we have:

**Theorem 5**. *If the holomorphic bisectional curvature is positive, then for any observable $F$ commuting with the Hamiltonian, the associated dispersion $V_t^F$ is a supermartingale, and* (4) *is a reduction process for $F$.*

In the linear theory, this result is intuitively expected, because the eigenstates of $H$ also diagonalise any commuting observable $F$. Indeed, when $\mathfrak{M} = \Gamma$ we have $\mathcal{K}_{FH} = \frac{1}{2}(1+\cos^2\theta)$, where $\theta$ is the angle between the vectors $\nabla^a H$ and $\nabla^a F$, from which it follows that $\frac{1}{2} < \mathcal{K}_{FH} \leq 1$, and thus reduction is guaranteed.

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